Stellar configurations in $f(R)$ theories of gravity

K. Henttunen,[*] T. Multamäki,[†] and I. Vilja[‡]
Department of Physics, University of Turku, FIN-20014 Turku, FINLAND

(Dated:)

We study stellar configurations and the space-time around them in metric $f(R)$ theories of gravity. In particular, we focus on the polytropic model of the Sun in the $f(R) = R - \mu^4/R$ model. We show how the stellar configuration in the $f(R)$ theory can, by appropriate initial conditions, be selected to be equal to that described by the Lane-Emden -equation and how a simple scaling relation exists between the solutions. We also derive the correct solution analytically near the center of the star in $f(R)$ theory. Previous analytical and numerical results are confirmed, indicating that the space-time around the Sun is incompatible with Solar System constraints on the properties of gravity. Numerical work shows that stellar configurations, with a regular metric at the center, lead to $\gamma_{PPN} \simeq 1/2$ outside the star *i.e.* the Schwarzschild-de Sitter -space-time is not the correct vacuum solution for such configurations. Conversely, by selecting the Schwarzschild-de Sitter -metric as the outside solution, we find that the stellar configuration is unchanged but the metric is irregular at the center. The possibility of constructing a $f(R)$ theory compatible with the Solar System experiments and possible new constraints arising from the radius-mass -relation of stellar objects is discussed.

I. INTRODUCTION

Current cosmological observations provide strong evidence against a critical density matter dominated universe. Observations on supernovae type Ia [1], cosmic microwave background [2] and large scale structure [3] all indicate that the expansion of the universe is not proceeding as predicted by general relativity (GR), if the universe is homogeneous, spatially flat and filled with non-relativistic matter. The underlying assumptions have hence been questioned, resulting in the emergence of a cosmological concordance model, the ΛCDM -model, describing a flat and homogeneous universe dominated by cold dark matter and dark energy in the form of a cosmological constant. Alternatives to the simple cosmological constant are, however, numerous (for a review see *e.g.* [4]) and other assumptions, such as *e.g.* homogeneity [5] have been questioned.

An alternative route to solving the dark energy problem is to consider modifying the underlying theory of gravity *i.e.* relaxing the assumption that general relativity is the correct theory of gravity on cosmological scales. A popular choice is the class of $f(R)$ gravity models that has received much attention in the recent literature (see *e.g.* [6, 7, 8, 9, 10, 11, 12, 13, 14] and references therein).

In a $f(R)$ gravity model, deviations from general relativity arise by generalizing the Einstein-Hilbert -action with an arbitrary function of the curvature scalar, $f(R)$. Such modification has to face many challenges that general relativity passes, including instabilities [15, 16, 17], solar system constraints(see *e.g.* [18, 19, 20] and references therein) and evolution of large scale perturbations [21, 22].

In particular, the solar system observations offer a good testing ground for any modification of general relativity by comparing the Parameterized Post-Newtonian (PPN) parameters [23, 24, 25, 26] with observations. This question has recently been extensively reviewed and discussed by a number of authors [27, 28, 29, 30] in order to determine the relevance of the Schwarzschild-de Sitter (SdS)-solution in the solar system (recently a new class of models that can evade the Solar System constraints has been introduced [31]). The SdS-metric is an exact vacuum solution in a large class of $f(R)$-theories of gravity that is in agreement with all solar system observations with an appropriate cosmological limit. However, the higher order nature of $f(R)$ theories makes the SdS-solution (see *eg.* [32, 33]) non-unique. This property of $f(R)$ theories also demonstrates itself on a cosmological scale, making reconstruction of the form of $f(R)$ from the expansion history of the universe non-unique [34].

As a result of the recent discussion, it has become clear that the SdS metric is unlikely to be the correct vacuum solution in the Solar System for the $1/R$ model. Instead, the PPN Solar System constraints are valid in a limit that corresponds to the limit of light effective scalar in the equivalent scalar-tensor theory. This is equivalent to requiring that one can approximate the trace of the field equations by Laplace's equation [28] in the corresponding $f(R)$ theory. This result has now also been considered by numerical calculations [35], where the field equations are integrated numerically from the center of a star for a fixed matter distribution.

Relatedly, in a recent work [36] we considered perfect static fluid sphere solutions in $f(R)$ theories of gravity. Again, the higher order nature of the $f(R)$ gravity theories demonstrates itself in that the mass distribution alone does not uniquely determine the gravitational theory, unless the boundary conditions are fixed. If one imposes the SdS-metric as a boundary condition, one finds that the solutions are constrained.

Here we consider these questions by numerical and an-

[*]Electronic address: kajohe@utu.fi
[†]Electronic address: tuomul@utu.fi
[‡]Electronic address: vilja@utu.fi

alytical means. We solve the set of field equations both inwards and outwards, *i.e.* by starting from the center and the boundary of a star. In contrast to [35], we do not fix the mass distribution beforehand and for completeness also study configurations with non-negligible pressure. Furthermore, we consider mass distributions with the SdS-metric as a boundary condition and discuss appropriate analytical limiting solutions near the origin.

II. $f(R)$ GRAVITY FORMALISM

The action for $f(R)$ gravity is

$$S = \int d^4x \sqrt{-g}\Big(\frac{1}{16\pi G} f(R) + \mathcal{L}_m\Big) \tag{1}$$

and the corresponding field equations derived by variating wrt the metric $g_{\mu\nu}$ are

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F(R) = 8\pi G T_{\mu\nu}. \tag{2}$$

Here $T_{\mu\nu}$ is the standard minimally coupled stress-energy tensor and $F(R) \equiv df/dR$. Contracting the field equations we get another useful form:

$$F(R)R - 2f(R) + 3\Box F(R) = 8\pi G(\rho - 3p). \tag{3}$$

We consider spherically symmetric, static configurations $(p = p(r),\ \rho = \rho(r))$ and adopt a metric:

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{4}$$

In this metric, the non-trivial component of the continuity equation $D_\mu T^{\mu\nu} = 0$ reads:

$$p' = -\frac{B'}{2B}(\rho + p), \tag{5}$$

where $' \equiv d/dr$. Note that, like in GR, the equation of continuity is automatically satisfied [37] and hence no additional information is gained on top of the field equations. On the other hand, one can choose the continuity equation as one of the equations to be solved instead of using the full set of field equations.

A. Equations

For a given action function $f(R)$, one can in principle take any suitable set of modified field equations, Eq. (2) along with the equation of state, $p = p(\rho)$, and solve for $\rho, A,\ B$. However, this can in practice prove to be problematic, since the modified Einstein's equations are highly non-linear and high order differential equations, up to fourth order in B and third order in A.

To simplify the problem, it is useful to consider $f(R)$ and $F(R)$ as independent functions of r. In order to correctly account for their dependence, one then needs to supplement the set of field equations with an appropriate additional constraint, $f = f(F)$, determined by the details of the particular $f(R)$ theory in question. In this description we are able to view $F(r),\ A(r),\ B(r)$ and ρ as the fundamental set of unknown functions to be solved. Note that unlike before, now the equation of continuity is not automatically satisfied but is an additional, independent differential equation. This is due to the fact that $F(r)$ (and $f(r)$) is not given in terms of the scalar curvature but is viewed just as a independent function of the radial coordinate r.

Thus the modified Einstein equation along with the equation of continuity forms the set of independent equations to be solved. These nonlinear equations are, however, only second order in F and B, and first order in A and ρ requiring in total six initial conditions for completely determining the solution. This is apparently one less that is needed if one proceeds by solving the field equations directly in terms of A and B, demonstrating that the higher derivatives of the modified field equations only appear in the combinations of the derivatives of the scalar curvature R.

In this paper, we consider the CDTT-model introduced in [6] with

$$f(R) = R - \frac{\mu^4}{R}, \tag{6}$$

but generalization to more complicated models is straightforward. In this particular case, it is easy to see that the equation relating f and F is

$$f = \mu^2 \frac{2 - F}{\sqrt{F - 1}}. \tag{7}$$

In a more general case, like $f(R) = R - \mu^4/R + R^2/\beta^2$, the functional relation is more involved and in general one may need to resort to numerical means.

As is well known, the CDTT-model has a homogeneous solution of de Sitter -metric with constant scalar curvature $R = R_0 \equiv -\sqrt{3}\,\mu^2$. In order to have the desired physically plausible late time behaviour, we therefore set $\sqrt{3}\,\mu^2 \sim H_0^2$.

Using Eq. (7) and defining $F(r) \equiv 4/3 + v(r),\ n(r) \equiv B'(r)/B(r)$, straightforward algebraic manipulations result in the following set of equations:

$$
\begin{aligned}
\frac{4n}{3r}+\frac{n^2}{3}-8G\pi A\rho+\frac{nv}{r}+\frac{n^2v}{4}+\frac{\mu^2A}{3\sqrt{\frac{1}{3}+v}}-\frac{\mu^2Av}{2\sqrt{\frac{1}{3}+v}}-\frac{nA'}{3A}-\frac{nvA'}{4A}+\frac{2n'}{3}+\frac{vn'}{2}-\frac{2v'}{r}+\frac{A'v'}{2A}-v'' &= 0\\
\frac{n^2}{3}+8G\pi Ap+\frac{n^2v}{4}+\frac{\mu^2A}{3\sqrt{\frac{1}{3}+v}}-\frac{\mu^2Av}{2\sqrt{\frac{1}{3}+v}}-\frac{4A'}{3rA}-\frac{nA'}{3A}-\frac{vA'}{rA}-\frac{nvA'}{4A}+\frac{2n'}{3}+\frac{vn'}{2}-\frac{2v'}{r}-\frac{nv'}{2} &= 0\\
\frac{4}{3r^2}-\frac{4A}{3r^2}+\frac{2n}{3r}+8G\pi Ap+\frac{v}{r^2}-\frac{Av}{r^2}+\frac{nv}{2r}-\frac{\mu^2Av}{2\sqrt{\frac{1}{3}+v}}+\frac{\mu^2A}{3\sqrt{\frac{1}{3}+v}}-\frac{2A'}{3rA}-\frac{vA'}{2rA}-\frac{v'}{r}-\frac{nv'}{2}+\frac{A'v'}{2A}-v'' &= 0\\
p'+\frac{n}{2}(p+\rho) &= 0.
\end{aligned}
$$

This set of equations is suitable for numerical integration made in this article.

Note, that the dependence on $B(r)$ dissapears completely from the equations (8), *i.e.* the equations are only first order in $n(r)$ reflecting the leftover free time scaling of the metric component $B(r)$.

B. Solution near the origin

In order to solve the field equations numerically from the center, one also has to consider the question of boundary conditions. Clearly, one cannot start the numerical integration from the origin $r=0$ due to singularities, but a small distance from it. Therefore one must select the initial values such that they correspond to a desired and a possible solution: otherwise, one might start the calculation from a point in parameter space that is unreachable by any solution that starts from the origin.

First we determine the asymptotically correct starting point by considering solutions corresponding to regular metrics at the origin. Thus we require $A(0)$, $n(0)$, $\rho(0)$ and $v(0)$ be finite and $p'(0)=0$. Moreover, the radial coordinate can be scaled so that $A(0)=1$ as usual. Expanding around the origin and solving the field equations gives, up to leading order:

$$
\begin{aligned}
A(r) &= 1+\frac{8G\pi\left(3p_0+2\rho_0\right)+\mu^2/\sqrt{\frac{1}{3}+v_0}}{12+9v_0}r^2\\
n(r) &= \frac{16G\pi\left(3p_0+2\rho_0\right)\sqrt{\frac{1}{3}+v_0}-\mu^2\left(2+3v_0\right)}{3\sqrt{\frac{1}{3}+v_0}\left(4+3v_0\right)}r\\
v(r) &= v_0+\left(\frac{4G\pi\left(3p_0-\rho_0\right)}{9}-\frac{\mu^2v_0}{6\sqrt{\frac{1}{3}+v_0}}\right)r^2, \quad (8)
\end{aligned}
$$

where the subscript 0 refers to the values at the origin. Pressure and density are constant at the level of the approximation. Regularity of the solution at origin leaves three free parameters v_0, p_0 and ρ_0, which completely determine the solution and hence the stellar structure. For a given equation of state $p=p(\rho)$ the number of parameters further reduces to two. Using these equations allows one to start the numerical integration a small distance from the origin while preserving the correct asymptoptic behaviour.

When one relaxes the requirement that the metric and $v(r)$ are to be finite at the origin, it is possible to find several mathematically plausible solutions at the vicinity of the origin with finite density $\rho(0)$ but having singular behaviour of metric. As we will see, such solutions arise easily whenever one tries to solve the field equations inwards, from the stellar boundary to the center.

C. Parametrisation of density and pressure

In order to consider more realistic matter distribution than predetermined toy models, we consider here polytropic stars *i.e.* stars with an equation of state $p=\kappa\rho^\gamma$. Here κ and $\gamma=1+1/n$ are constants and n is often referred to as the polytropic index. Such equations of state are useful in studying white dwarfs, neutron stars and can also be used as a simple model of main sequence stars such as the Sun [38]. For a polytropic equation of state, one can straightforwardly solve the continuity equation, Eq. (5):

$$
\rho(r)=\kappa^{1/(1-\gamma)}\Big(\big(\frac{B(r)}{\bar{B}}\big)^{(1-\gamma)/(2\gamma)}-1\Big)^{1/(\gamma-1)}. \quad (9)
$$

Requiring, that ρ vanishes at the stellar surface $r=r_R$, where $\bar{B}=B(r_R)$, sets $\gamma>1$. (It can be shown that requiring finite radius constrains $\gamma>6/5$ in the Newtonian Lane-Emden -model [39].) Similarly, since ρ' is also vanishing at the boundary [36], we must further require that $\gamma<2$.

For numerical work, it is advantageous to use scaled variables, $\rho=\rho_0\theta^{1/(\gamma-1)}$, $r=\alpha x$, $\alpha=\sqrt{\kappa\gamma/(4\pi G(\gamma-1))}\rho_0^{(\gamma-2)/2}$ (see eg. [39]). In these variables, the fundamental equation of a Newtonian star reduces to the Lane-Emden -equation. In the case of GR or modified gravity, this is not the case, but the same change of variables is still useful. Using θ instead of ρ is

advantageous also due to the fact that unlike ρ', θ' does not vanish at the boundary of the star, making identification of the star's surface easier.

III. NUMERICAL RESULTS

Our next task is to compute numerically stellar profiles for certain polytropic cases for the $f(R) = R - \mu^4/R$ model. We have done this for a number of polytropic equations of state, both starting from the center of the star and from the boundary. The computed profiles θ as well as metric components, A and n, can then be compared to corresponding functions of Newtonian polytropes determined by the Lane-Emden -equation. Note that we require that stellar solutions have finite radii, unlike *e.g.* toy models where the profile is approximated by an exponential function.

We consider the Sun, as a representative of main sequence stars, white dwarfs as well as neutron stars. The polytropic model of the Sun, with $\gamma_\odot = 1.2985$, $\rho_0 = 1.53 \times 10^5\,\mathrm{kg\,m^{-3}}$ and $p_0 = 3.00 \times 10^{16}\,\mathrm{N\,m^{-4}}$ gives a fair approximation to the Stellar Standard Model [40]. For the relativistic white dwarfs, we use $\gamma = 4/3,\ \kappa = 1.4 \times 10^{-7}(\mathrm{kg\,m^{-3}})^{-1/3}$ and for nonrelativistic neutron stars $\gamma = 5/3,\ \kappa = 3.5 \times 10^{-11}(\mathrm{kg\,m^{-3}})^{-2/3}$[39].

A. Solution starting from the center outwards

Using Eqs (8) as a starting point, we can now integrate numerically the field equations along with the structural equation, Eq. (7), for a given stellar model. Integration is stopped at the stars surface *i.e.* when $\theta = 0$. Fixing the central density, ρ_0, fixes also p_0 via the polytropic equation of state but v_0 remains as a free parameter. In Fig. 1 we show the density profile for the Sun for a range of values of v_0. We see how changing v_0 scales the profile so that a larger value of v_0 leads to a star with a larger radius and vice versa. The corresponding Lane-Emden -solution *i.e.* the polytropic model of the Sun is equal to choosing $v_0 = 0$ with very high precision. In Fig. 2 we show the evolution of v as a function of the scaled radial distance x. Two properties are notable: the evolution is very small *i.e.* the value changes very little over the radius of the star and the value is monotonically decreasing. The latter property is important when we consider fitting the star to a SdS-spacetime ($v = 0$) outside the star.

Although the density profile in general closely resembles the Newtonian one, the behaviour of the metric is completely changed. Fitting the general metric to the PPN SdS-solution,

$$\begin{aligned} B(r) &= 1 - \frac{2GM}{r} - H^2 r^2 \\ A(r)^{-1} &= 1 - \gamma_{PPN}\frac{2GM}{r} - H^2 r^2, \end{aligned} \qquad (10)$$

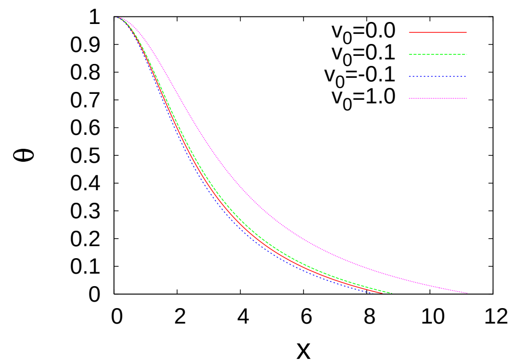


FIG. 1: Density profiles of the polytropic model of the Sun for different values of v_0.

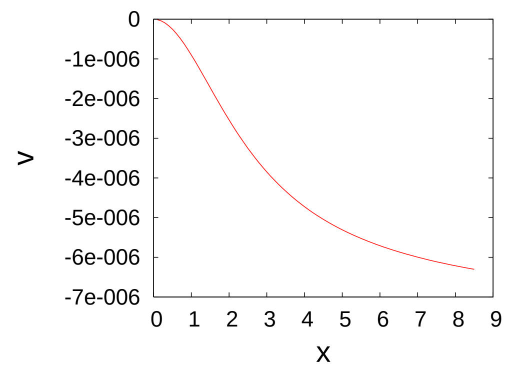


FIG. 2: Evolution of $v(r) = F(r) - 4/3$ for the Sun, $v_0 = 0,\ r = \alpha x$.

we can solve the PPM-parameter γ_{PPN} at distance r as

$$\gamma_{PPN} = \frac{1 + \frac{B(r)}{rB'(r)}}{1 - \frac{A(r)}{rA'(r)}}. \qquad (11)$$

Note that here we have neglected the H^2r^2 -term as the cosmological parameter is extremely tiny compared to relevant solar system scales.

Numerical work shows, that at the stellar boundary r_R γ_{PPN} tends to be near value $\gamma_{PPN} = 0.5$, with small variations depending on the value v_0. In Fig. 3 we show the evolution of γ_{PPN} outside the Sun for the $v_0 = 0$ case but other choices of v_0 give essentially identical results. The corresponding physical distance is equal to $r \approx x \times 8 \times 10^8$ m. From the figure we see that the spacetime outside the Sun in the CDTT-model is in gross violation of the experiments, remaining very close to 1/2 far outside the Solar System. The behaviour at very large x is explained by the fact that we ignore the cosmological term in the calculation of γ_{PPN}, Eq. (11). The effect of the cosmological term should be included when $2GM/r \sim H_0^2 r^2 \sim \mu^2 r^2$, *i.e.* $x \sim 10^{11}$, in good agreement with the numerical calculation.

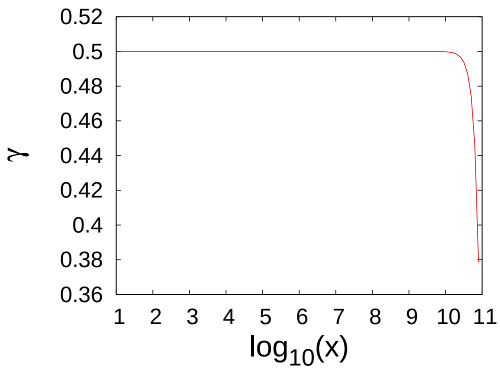


FIG. 3: Evolution of the PPN-parameter γ_{PPN} outside the Sun.

Note that the scale l_{PPN} at which we expect the value of γ to approach unity, *i.e.* the range of the effective scalar in corresponding scalar-tensor theory (see *e.g.* [14, 23, 24] and reference therein), is $l_{PPN} \sim 1/(\alpha\mu) \sim 10^{18}$ which far beyond the scale at which the cosmological term becomes effective. In the context of this particular $f(R)$ model, if one wishes to have $\gamma_{PPN} \to 1$ on scales where we can ignore the cosmological term, *i.e.* $2GM/r \gg \mu^2 r^2$ when $r \sim 1/\mu$, sets $\mu \gtrsim 1/\text{km}$ for the Sun, $M \approx 2 \times 10^{30}$ kg. This shows how in this model we cannot choose the different relevant scales to have physically meaningful values.

We have performed corresponding calculations also for relativistic white dwarfs and nonrelativistic neutron stars. Although stellar numerical values and scales are clearly different, qualitative conclusions remain: density profiles are close to Newtonian ones whereas metric profiles are completely different and $\gamma_{PPN} \approx 0.5$. We have also considered configurations with pressure comparable to density in which case we find that γ_{PPN} can deviate significantly from 1/2.

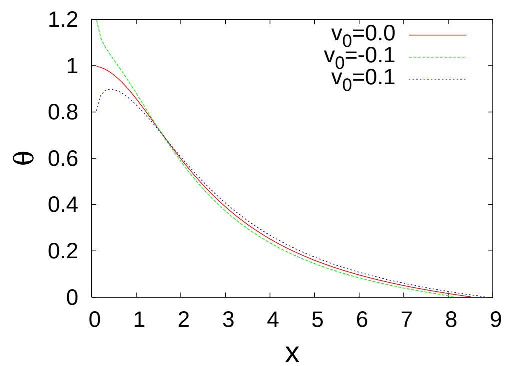
B. Solutions with the SdS metric as the external solution

We have seen that starting with a regular solution at the origin leads to unacceptable space-time outside the star, in good agreement with previous work [27, 28, 29, 30, 35]. This result suggests that starting with a physically acceptable outside solution, the SdS-metric, will lead to an irregular solution at the center of the star.

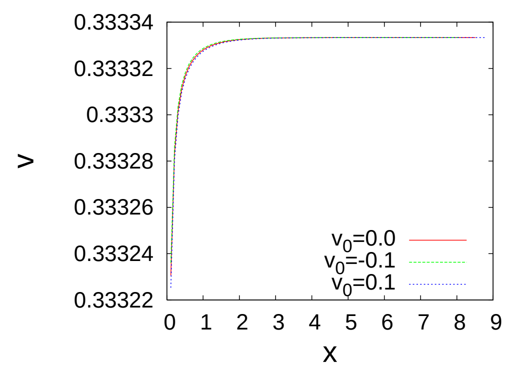
In order to determine the relevance of the SdS-metric for the particular $f(R)$ model in question, we solve Eqs (8) starting from the stellar boundary at given radius r_R. We set the boundary conditions by requiring that external metric is the SdS-metric

$$B_{ext}(r) = A_{ext}(r)^{-1} = 1 - \frac{2GM}{r} - H^2 r^2, \qquad (12)$$

where $H^2 = \mu^2/(4\sqrt{3})$. At the boundary we require that $n(r_R) = B'_{ext}(r_R)/B_{ext}(r_R)$, $A(r_R) = A_{ext}(r_R)$ as well as $v(r_R) = v'(r_R) = \theta(r_R) = 0$, corresponding to the SdS-metric [36]. In practice we first fix central density $\rho(0)$, and then set the gravitating mass M and stellar radius r_R, by solving equations from the center outwards for a given v_0. The computed mass and radius are then used as parameters in the outside metric, which then fixes the boundary condition for n and A. In Fig. 4(a), we show the stellar profiles corresponding to different choices of v_0 *i.e.* corresponding to different values of M and r_R. From the figure we see that for $v_0 = 0$, or the Sun, the density profile is unchanged *i.e.* fixing the outside solution to the SdS-metric leads to a physically acceptable density profile. When v_0 deviates from zero, the solution diverges near the origin. In the same figure we also show



(a) Density profiles.



(b) Profiles of $v(r)$, $r = \alpha x$.

FIG. 4: Stellar configurations with external SdS-metric.

the corresponding evolution of v, Fig. 4(b). In all of the cases the conclusion is the same: v diverges at the origin. This is as expected since we have already seen that any solution with a regular metric and density profile at the origin leads to a experimentally unacceptable outside solution.

Requiring the SdS-solution as the outside metric leads to a divergent v and hence the scalar curvature diverges at the origin. Similarly also the metric components A and n, are irregular at the origin. In summary, if one relaxes

the requirement of regularity of the metric at the origin, one can have the SdS-metric as the outside solution and reproduce the density profile.

It is worth noting, that if the star does not follow polytropic equation of state, especially in the core of the star, non-singular solution with external SdS-metrics can be found when $v_0 = 0$. In particular the regularity of the scalar curvature is seen directly from Eq. (3) if the equation of state near the center of the star is relativistic $p = \rho/3$. We have confirmed this phenomenon by using an equation of state, which is polytropic in the outer region and relativistic in the core. This resembles the case of massive, relativistic neutron stars [39].

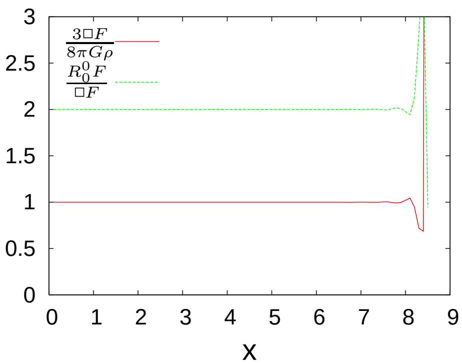


FIG. 5: Comparison of the size of different terms of the field equations for the Sun.

IV. COMPARISON WITH GENERAL RELATIVITY

We have seen that the case $v_0 = 0$ reproduces the result from using the Lane-Emden -equation very well. We can understand this behaviour analytically by considering the field equations in GR and in the f(R) theory.

In GR, the field equations can be written in the form $R^\nu_\mu = 8\pi G(T^\nu_\mu - g^\nu_\mu T/2)$, where $T = T^\mu_\mu$. In the limit of negligible pressure, the 00-component reads as $R^0_0 \approx 4\pi G\rho$. The scalar curvature, R, follows the density *i.e.* $R \sim \rho$.

In the CDTT-model, the situation is different. Now, as numerical calculations show, inside and outside the star, $f(R) \sim R \sim \mu^2$ and $F \sim \mathcal{O}(1)$. From the field equations one would then expect that for the 00-component, the FR^0_0 and $\Box F$ terms to be dominant *i.e.* $FR^0_0 + \Box F \approx 8\pi G\rho$ or if one considers the contracted equation, Eq. (3), $3\Box F \approx 8\pi G\rho$. In Fig. 5 we plot the relevant terms for the Sun (here $v_0 = 0$, but the situation is unchanged if v_0 is varied as long as $F \gg f$). From the figure we see that inside the star the approximations hold and hence, we can write $FR^0_0 \approx 16\pi G\rho/3$. From the numerical work we furthermore know, that the value of v or F changes very little inside the star and hence we can approximate $F \approx const. = F_0 = 4/3 + v_0$. We have then

$$R^0_0 \approx \frac{16\pi G}{3F_0}\rho \approx \frac{4\pi G}{1 + \frac{3}{4}v_0}\rho. \tag{13}$$

Comparing to the GR expression, we see that there exists a simple scaling between the two expressions. In the Lane-Emden -equation this scaling signals its presence in the scaling of the radial coordinate, $r = \alpha x \propto x/\sqrt{G}$, if ρ_0 is fixed *i.e.* we expect that

$$\theta_{f(R)}(x) \approx \theta_{GR}\left(\frac{x}{\sqrt{1 + \frac{3}{4}v_0}}\right). \tag{14}$$

In Fig. 6 we show the Sun's profile with $v_0 = 1$ (solid red line) along with the Lane-Emden -solution $\theta_{LE}(x)$ (dotted green line) and a scaled Lane-Emden -solution, $\theta_{LE}(x/\sqrt{1 + (3/4)v_0})$. The line at the bottom is the difference between the scaled Lane-Emden -solution and the numerical solution for $v_0 = 1$ magnified by a factor of 10^5. As we can see, the scaled solution reproduces the numerical solution very well.

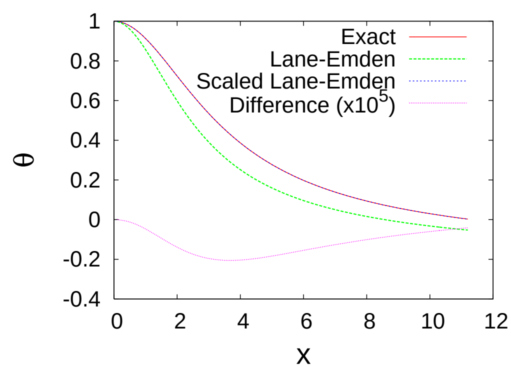


FIG. 6: Comparison of exact $f(R)$-model, Lane Emden and scaled Lane-Emden -model density profiles.

One can hypothesize that a similar scaling would exist in a general $f(R)$ theory, $f(R) = R + g(R, \mu)$. In order for such a theory to explain late time acceleration, the extra terms will in general have a new scale μ associated with the value of the present Hubble parameter (in principle one could also consider very finely tuned theories, where different parameters with different scales would conspire to produce current acceleration). The vacuum state of the theory is such that it possesses non-zero constant curvature, $R_0 \sim \mu^2$. Examples of such theories are $f(R) = R - \mu^4/R$ and $f(R) = R - \mu^4/R + R^2/\beta^2$. In the latter case, choosing $\beta \sim \mu$ helps to avoid Solar System constraints [28] by making the mass of effective scalar large. If R remains close to the cosmological value inside the star, like in the CDTT-model, it is natural to expect that $f \sim R \sim \mu^2 \sim H_0^2$ and $F \sim \mathcal{O}(1)$ and hence a similar scaling property should apply.

V. CONCLUSIONS

In the present paper we have analyzed the properties of polytropic stars in a generalized gravity model. In particular we have considered the $f(R)$ model with $f(R) = R - \mu^4/R$ with the conclusion that the density profiles in general resemble the Newtonian Lane-Emden -solutions. Requiring that stellar solution is regular at the origin, we found that slightly varying the central curvature, v_0, the stellar mass and radius are changed but preserve their functional Lane-Emden -form. However, the metric components are drastically different from the Lane-Emden -case and therefore our results for the external metric conform to previously calculated results for completely pressure-less matter [28, 35]. In particular the PPN parameter γ_{PPN} outside the start is near $\gamma_{PPN} = 1/2$.

If we do not require complete regularity of stellar solution at the origin, but assume the external SdS-solution, we still find stellar profiles in good agreement with the Lane-Emden solution. Differences appear only near the center of the star and these deviations depend on how much the mass and radius differ from the corresponding Newtonian configuration. The interior solution for metric components and curvature for such stars are always singular, although stars with relativistic matter at the core may evade this property.

Consequently, the $f(R) = R - \mu^4/R$ model is not experimentally suitable to describe the space time around the Sun. A possible way out is to relax the requirements set for the central boundary conditions, but a more plausible approach is to modify the functional form of $f(R)$. The form of the action function $f(R)$ should differentiate the cosmological Hubble scale $R_0 \sim H_0^2$ determined by $2f(R_0) = R_0 f'(R_0)$ and the effective scalar mass scale $\propto 1/f''(R_0)$. Then it may be possible to have $\gamma_{PPN} \to 1$ at a distance l_{PPN} small compared to solar system distances, possibly redeeming some of the $f(R)$ models. This is, however, not possible in the $f(R) = R - \mu^4/R$ model, because both scales are controlled by a single parameter μ, which when set to the cosmologically relevant value $\mu \sim H_0$, leads to $l_{PPN} \gg 1/H_0$. In other words, the asymptotic SdS-metric is never reached.

Seeking possible ways to save $f(R)$ models note also, that as discussed in [36], a suitable choice of $f(R)$ may exactly reproduce GR/Newtonian density profiles changing only A and v. In this case we do not, however, know much about their behaviour neither outside the star nor near the center of the star and even cosmological constraints are unknown.

Possibly several more general $f(R)$ models are physically acceptable, but in particular the model with $f(R) = R - \mu^4/R + R^2/\beta^2$ may do after considerable fine-tuning of β. Note however, that in this case an important distinction compared to the CDTT-model applies. In the CDTT-model, $F_0 \approx 4/3$ or $v_0 \approx 0$, inside and outside the star so that the field equations are effectively similar to the GR counteparts, Eq. (13). When the R^2 term is added to the action, $F_0 \neq 4/3$ outside the star and hence if we still wish to have $F_0 \approx 4/3$ inside star in order to reproduce the Lane-Emden -solution, F must evolve significantly over the radius of the star. Otherwise, if F_0 remains approximately constant, either the SdS-solution is not the correct outside solution, or the radius of the star will be different than in GR. This argument potentially offers a new, general constraint on $f(R)$ theories of gravity, motivating further work.

Acknowledgments

We thank Kimmo Kainulainen, Vappu Reijonen and Daniel Sunhede as well as Chris Flynn for valuable discussions during this work. KH is supported by the Wihuri foundation. TM gratefully acknowledges support from the Academy of Finland.

[1] A. G. Riess *et al.*, Astron. J. **116**, 1009 (1998); S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999); Astron. Astrophys. **447**, 31 (2006) [arXiv:astro-ph/0510447].
[2] D. N. Spergel *et al.* , Astrophys. J. Suppl. **148**, 175 (2003); D. N. Spergel *et al.*, arXiv:astro-ph/0603449.
[3] M. Tegmark *et al.*, Phys. Rev. D **69**, 103501 (2004); G. Efstathiou *et al.*, Mon. Not. Roy. Astron. Soc. **330**, L29 (2002).
[4] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989); P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003) [arXiv:astro-ph/0207347].
[5] H. Alnes, M. Amarzguioui and O. Gron, Phys. Rev. D **73**, 083519 (2006) [arXiv:astro-ph/0512006].
[6] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D **70**, 043528 (2004).
[7] S. M. Carroll, A. De Felice, V. Duvvuri, D. A. Easson, M. Trodden and M. S. Turner, Phys. Rev. D **71**, 063513 (2005).
[8] G. Allemandi, A. Borowiec and M. Francaviglia, Phys. Rev. D **70**, 103503 (2004).
[9] X. Meng and P. Wang, Class. Quant. Grav. **21**, 951 (2004).
[10] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003).
[11] S. Nojiri and S. D. Odintsov, Phys. Lett. B **576**, 5 (2003).
[12] S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002).
[13] R. P. Woodard, arXiv:astro-ph/0601672.
[14] S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007) [arXiv:hep-th/0601213].
[15] A. D. Dolgov and M. Kawasaki, Phys. Lett. B **573**, 1 (2003).
[16] M. E. Soussa and R. P. Woodard, Gen. Rel. Grav. **36**, 855 (2004) [arXiv:astro-ph/0308114].
[17] V. Faraoni, Phys. Rev. D **72**, 124005 (2005) [arXiv:gr-qc/0511094].
[18] T. Chiba, Phys. Lett. B **575**, 1 (2003).

[19] E. E. Flanagan, Class. Quant. Grav. **21**, 417 (2003); Class. Quant. Grav. **21**, 3817 (2004); G. Magnano and L. M. Sokolowski, Phys. Rev. D **50**, 5039 (1994).
[20] T. Clifton and J. D. Barrow, Phys. Rev. D **72**, 103005 (2005) [arXiv:gr-qc/0509059].
[21] R. Bean, D. Bernat, L. Pogosian, A. Silvestri and M. Trodden, Phys. Rev. D **75**, 064020 (2007) [arXiv:astro-ph/0611321].
[22] Y. S. Song, W. Hu and I. Sawicki, Phys. Rev. D **75**, 044004 (2007) [arXiv:astro-ph/0610532].
[23] T. Damour and G. Esposito-Farese, Class. Quant. Grav. **9**, 2093 (1992).
[24] G. Magnano and L. M. Sokolowski, Phys. Rev. D **50**, 5039 (1994) [arXiv:gr-qc/9312008].
[25] G. J. Olmo, Phys. Rev. Lett. **95**, 261102 (2005) [arXiv:gr-qc/0505101].
[26] S. Capozziello, A. Stabile and A. Troisi, arXiv:gr-qc/0603071; G. Allemandi, M. Francaviglia, M. L. Ruggiero and A. Tartaglia, Gen. Rel. Grav. **37**, 1891 (2005) [arXiv:gr-qc/0506123]; T. P. Sotiriou, Gen. Rel. Grav. **38**, 1407 (2006) [arXiv:gr-qc/0507027].
[27] A. L. Erickcek, T. L. Smith and M. Kamionkowski, Phys. Rev. D **74**, 121501 (2006) [arXiv:astro-ph/0610483].
[28] T. Chiba, T. L. Smith and A. L. Erickcek, arXiv:astro-ph/0611867.
[29] X. H. Jin, D. J. Liu and X. Z. Li, arXiv:astro-ph/0610854.
[30] T. Faulkner, M. Tegmark, E. F. Bunn and Y. Mao, arXiv:astro-ph/0612569.
[31] W. Hu and I. Sawicki, arXiv:0705.1158 [astro-ph].
[32] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, JCAP **0502**, 010 (2005) [arXiv:hep-th/0501096].
[33] T. Multamaki and I. Vilja, Phys. Rev. D **74**, 064022 (2006) [arXiv:astro-ph/0606373].
[34] T. Multamaki and I. Vilja, Phys. Rev. D **73**, 024018 (2006) [arXiv:astro-ph/0506692].
[35] K. Kainulainen, J. Piilonen, V. Reijonen and D. Sunhede, arXiv:0704.2729 [gr-qc].
[36] T. Multamaki and I. Vilja, arXiv:astro-ph/0612775.
[37] T. Koivisto, Class. Quant. Grav. **23**, 4289 (2006) [arXiv:gr-qc/0505128].
[38] A. S. Eddington, *The Internal Constitution of the Stars*, Cambridge, (1930); S. Chandrasekhar, *An introduction to the study of stellar structure*, Dover, (1939).
[39] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley & Sons, (1972).
[40] A.. Hendry, Am. J. Phys. **61** (10) (1993).